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Endogenous Market Size, Partial Privatization, and Urban Structure

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Abstract

This paper integrates a mixed duopoly market into an open monocentric city model and develops a general equilibrium framework in which market size is endogenously determined through urban structure and household location behavior. A public firm and a private firm compete in the manufacturing sector located at the central business district.

We analyze how public firm privatization affects the equilibrium price of manufactured goods and the equilibrium city size. Privatization increases the equilibrium price and reduces city size. However, once indirect effects through city size are taken into account, the welfare impact of privatization becomes ambiguous. As a result, the conclusion of Matsumura (1998) that partial privatization always maximizes social welfare does not generally hold. Numerical examples illustrate that full nationalization or full privatization can be optimal depending on parameter values.

Keywords: Partial Privatization; Mixed Duopoly; Open City Model; Endogenous Market Size; General Equilibrium
JEL Classification: H42; R13; L13

1 Introduction

This paper constructs a model that integrates a mixed duopoly market into a small open monocentric city framework. We consider an economy consisting of manufactured goods and agricultural goods produced in the central

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business district (CBD), which is located at the edge of the city. The former is characterized by a mixed duopoly market in which a public firm and a private firm produce manufactured goods, while the latter is produced under a constant returns to scale technology and is used as the numeraire.

Mixed oligopoly (duopoly) models in which a welfare-maximizing public firm and a profit-maximizing private firm compete have been extensively studied in the industrial organization literature, beginning with the seminal work of De Fraja and Delbono [5]. A theoretical analysis of mixed oligopoly markets has been the subject of numerous studies. De Fraja and Delbono [5] show that the privatization of a public firm is desirable in terms of social welfare when the number of private firms is sufficiently large. As for any policy in the mixed oligopoly market, White [21] show that social welfare remains unchanged before and after privatization when a production subsidy is introduced. Matsumura [13] extends the framework by introducing partial privatization of the public firm and demonstrates that partial privatization is socially optimal. Moreover, Tomaru [20] demonstrates that the optimal production subsidy is independent of the public ownership share of the partially privatized firm within the framework of Matsumura [13], and that both output levels and profits of all firms are unaffected under the optimal subsidy.

Some studies focus on the relationship between location or region and mixed oligopolistic markets. Cremer *et al.*[4], Inoue *et al.*, [9], and Matsushima and Matsumura[14] extend the mixed oligopoly framework by incorporating Hotelling's [8] linear city model and analyze spatial competition, discussing the equilibrium number of public and private firms and their equilibrium locations. However, these studies abstract from migration between regions or within a city, since they investigate the location strategies of public and private firms under mixed oligopolistic competition. Although Naito [17] analyzes a Harris–Todaro [7] type dual economy with household migration between urban and rural regions and investigates the effects of public firm privatization on urban unemployment under mixed duopoly, the analysis abstracts from urban structure by treating both regions as points.

On the other hand, Alonso [3], Muth [16], and Mills [15] examine household residential location choices within a monocentric city. A monocentric city is characterized by a single central business district (CBD), to which all households commute for work. Production sectors are located exclusively in the CBD, while no production activity exists elsewhere in the city. Households are assumed to have employment opportunities only in the CBD, and choose their residential locations by trading off commuting costs to the CBD against land rents. As the monocentric city model focuses on the residential behavior of households within a city, interregional migration is not modeled. Thus, the monocentric city model focuses on the equilibrium residential location of households within a city and, like Hotelling [8], omit from interregional migration.

Some studies develop models that simultaneously consider interregional migration and residential equilibrium within a city. Abdel-Rahman [1], Krugman and Elizondo [11], and Tabuchi [19] integrate a core-periphery model with a monocentric city framework and elucidate the mechanism driving interregional migration. They all combine a monopolistic competition model developed by Dixit and Stiglitz [6] with a monocentric city model to describe household migration across regions. Although these models are new economic geography models based on Krugman [10], they adopt monopolistic competition instead of mixed oligopoly market structures.

Therefore, we introduce a mixed duopoly market located at the CBD into the production sector of an open monocentric city model. Although Naito [18] considers a setting in which the production sector facing a mixed duopoly market is located at the CBD, the analysis does not take partial privatization of the public firm into account, as in Matsumura [13]. Accordingly, we extend Naito [18] by introducing partial privatization of the public firm into the mixed duopoly market and investigate how progressive privatization affects the equilibrium price of manufactured goods, the equilibrium city size, and social welfare. Therefore, the contributions of this paper are as follows. First, we combine a mixed duopoly market with an open city model and construct a general equilibrium framework in which the scale of the manufactured goods market, measured by the number of households in a city, is endogenously determined. As a result, our model determines the market size endogenously, which has typically been treated as exogenous in the mixed oligopoly literature, through urban structure and households' residential location behavior. Second, we examine analytically how public firm privatization affects the equilibrium price of manufactured goods and the equilibrium city size. As a result of our analysis, we show that public firm privatization increases the equilibrium price of manufactured goods and shrinks the equilibrium city size. Finally, we demonstrate that when market size is determined endogenously, the result of Matsumura [13], which states that partial privatization always maximizes social welfare, does not hold in general. Namely, when the indirect effects through city size are taken into account, the optimal level of public firm privatization depends on parameter values, and in some cases either full nationalization or full privatization can be optimal.

The remainder of this paper is organized as follows. Section 2 presents the basic model, detailing household behavior, urban structure, and the production sector, especially the manufactured goods sector characterized by a mixed duopoly market, and demonstrates the uniqueness of the equilibrium population size. Section 3 conducts a comparative statics analysis to clarify the effects of public firm privatization on the equilibrium. The final section summarizes the main results and discusses directions for future research.

2 The model

We consider an open monocentric city model with a linear spatial structure. We assume that the central business district (CBD) is located at the edge of the city. Let r denote a location within the city. Since the CBD is assumed to be located at the edge of the city, it is represented by $r = 0$. All households residing in the city must commute to the CBD to earn income, as employment opportunities exist only at the CBD. When commuting to the CBD, households incur a commuting cost of t per unit of distance. Households are endowed with one unit of labor, which they supply inelastically to the production sector in the city. Regarding urban land, we assume public land ownership. As a result, aggregate land rents generated in the city are redistributed to households.

2.1 Households

We assume that all households have identical preferences; that is, they share the same utility function and derive utility from consuming manufactured goods and an agricultural good, which serves as the numeraire, at location r measured from the CBD. Following Naito [18], the utility function is specified as follows.

$$U(x, y; r) = ax - \frac{1}{2}x^2 + y, \tag{1}$$

where a, x, and y denote the reservation demand for the manufactured good, the consumption of the manufactured good, and the consumption of the agricultural good, respectively. This quasi-linear specification allows us to obtain closed-form solutions while keeping the strategic interaction in the mixed duopoly transparent. Households residing in the city consume one unit of land inelastically. Since households incur a commuting cost of t per unit of distance between the CBD and location r, and aggregate land rents are redistributed to them, let $R(r)$ denote the land rent per unit of land at location r. The budget constraint of a household residing at r is given by

$$w + \frac{1}{N}\int_0^{r_f} R(r)\, dr = px + y + tr + R(r), \tag{2}$$

where w, p, $R(r)$, and N denote the wage rate, the price of manufactured goods, the land rent at location r, and the number of households in the city, respectively. Maximizing (1) subject to (2), we derive the demand function for manufactured goods of a household residing at location r as follows.

$$x = a - p \tag{3}$$

Since the utility function specified in (1) is quasi-linear, the demand function exhibits no income effect. Substituting (3) into (2) yields the demand for agricultural goods:

$$y = w + \frac{1}{N}\int_0^{r_f} R(r)dr - R(r) - tr - p(a-p) \tag{4}$$

Moreover, substituting (3) and (4) into (1) yields the indirect utility function of a household residing at location r.

$$u(r) = \frac{1}{2}(a-p)^2 + w + \frac{1}{N}\int_0^{r_f} R(r)dr - R(r) - tr \tag{5}$$

Because the demand function for manufactured goods is linear, as shown in (3), the consumer surplus of a household residing at location r, denoted by $CS(r)$, is given by

$$CS(r) = \frac{1}{2}(a-p)^2 \tag{6}$$

Since the total number of households in a city is given by N, the total demand for manufactured goods in a city $Q(p)$ is given by

$$Q(p) = N(p)x(p) = N(a-p) \tag{7}$$

2.2 Urban structure

In the previous subsection, we derived household demand for manufactured and agricultural goods, along with the indirect utility function and consumer surplus, as functions of the manufactured-goods price p and the land rent schedule $R(r)$. Here, we consider the urban equilibrium to determine land rent and the urban boundary endogenously. Since the indirect utility function of a household in the city is given by (5), it must be equal to $\bar{U}$ in equilibrium, that is,

$$\bar{U} = \frac{1}{2}(a-p)^2 + w + \frac{1}{N}\int_0^{r_f} R(r)dr - R(r) - tr \tag{8}$$

Assuming that the opportunity cost of land is zero, the land rent at the city boundary r_f, denoted by $R(r_f)$, is equal to zero. Thus, by substituting $r = r_f$ into (5), we obtain the utility of a household residing at the urban boundary.

In urban equilibrium, households within the city have no incentive to relocate to other locations because they attain the same level of utility. Equating the equilibrium utility of households in the city with that of a household residing at the urban boundary, the following condition must hold:

$$\bar{U} = \frac{1}{2}(a-p)^2 + w + \frac{1}{N}\int_0^{r_f} R(r)\,dr - tr_f. \tag{9}$$

Under the linear city model, the assumption that the supplied lot size is one at each location and that households consume land inelastically implies that r_f equals N. By subtracting (8) from (9) and taking into account that $r_f = N$, the equilibrium land rent function is obtained as follows.

$$R(r) = t(N-r) \tag{10}$$

From (10), the equilibrium land rent is a decreasing function of the distance from the CBD, denoted by r, and an increasing function of the unit commuting cost to the CBD, denoted by t. Therefore, using (10) and $r_f = N$, we derive the aggregate land rent, denoted by TLR, as follows.

$$TLR = \int_0^{r_f} R(r)dr = \frac{t}{2}N^2 \tag{11}$$

Substituting (11) into (8) and solving for r_f, we derive the equilibrium urban boundary and population as follows.

$$r_f^* = N^* = \frac{1}{t}\left[(a-p)^2 + 2(w-\bar{U})\right] \tag{12}$$

From (12), we see that the equilibrium urban boundary and population depend on the price of manufactured goods and the wage rate. Although most traditional monocentric city models treat the price of goods or income (i.e., the wage rate) as exogenous parameters, these variables are determined endogenously in our model.

2.3 Production

Next, we consider the production sectors in the model. We assume that both manufactured goods and agricultural goods are produced in the central business district (CBD) and that labor is the only input. The agricultural good is produced under constant returns to scale, requiring one unit of labor to produce one unit of output. Since the agricultural good is taken as the numeraire, the wage rate of labor employed in this sector is normalized to one. Assuming that households can move freely between sectors, the wage rate in the manufactured goods sector is also equal to one; that is, $w = 1$.

Moreover, in the manufactured goods sector, we assume that firms face a mixed duopoly market in which a public firm competes with a private firm. Following Matsumura [13], the public firm is assumed to maximize a weighted average of social welfare and its own profit, whereas the private firm maximizes profit. Let q_0 and q_1 denote the output levels of the public firm and the private firm, respectively. Both firms are assumed to take the city population N as given. Given the total demand for manufactured goods in the city in (7), the inverse demand function is given by

$$p = a - \frac{1}{N}(q_0 + q_1) \tag{13}$$

The public firm and the private firm use only labor as an input. Specifically, they require αq_0 and q_1 units of labor to produce q_0 and q_1, respectively. The quadratic cost structure captures increasing marginal costs. Due to free mobility of labor between sectors, the wage rate is equal to one. Therefore, the cost function of firm i, denoted by $C_i(q_i)$, is given by

$$C_i(q_i) = \begin{cases} \frac{\alpha}{2}(q_i)^2 & (i = 0) \\ \frac{1}{2}(q_i)^2 & (i = 1) \end{cases} \tag{14}$$

where α represents the productivity gap between the public firm and the private firm. We assume that the public firm is less efficient than the private firm; that is, $\alpha > 1$. The profit functions of the public firm, denoted by π_0, and the private firm, denoted by π_1, are given as follows.

$$\pi_0 = \left(a - \frac{1}{N}(q_0 + q_1)\right) q_0 - \frac{\alpha}{2} q_0^2 \tag{15}$$

$$\pi_1 = \left(a - \frac{1}{N}(q_0 + q_1)\right) q_1 - \frac{1}{2} q_1^2 \tag{16}$$

Since the public firm maximizes a weighted average of social welfare and its own profit, we need to define a social welfare function in the model. In a monocentric city model, if social welfare is defined as the Benthamite sum of the utilities of households residing on the interval $r \in [0, r_f]$, social welfare would mechanically increase with population size. To avoid this implication, we define social welfare as the sum of consumer surplus, producer surplus, and aggregate land rent, net of $N\bar{U}$. Therefore, social welfare in the city is given by

$$\begin{aligned} W &\equiv CS + PS + \int_0^{r_f} R(r)\,dr - N\bar{U} \\ &= \int_0^Q p(q)\,dq - \sum_{i\in\{0,1\}} C_i(q_i) \ + \int_0^{r_f} R(r)\,dr \ - \ N\bar{U} \\ &= a(q_0+q_1) - \frac{(q_0+q_1)^2}{2N} - \frac{\alpha}{2}q_0^2 - \frac{1}{2}q_1^2 + \frac{t}{2}N^2 - N\bar{U}. \end{aligned} \tag{17}$$

Although Naito [18] considers a mixed duopoly market in which a public firm and a private firm are located in the CBD and maximize social welfare and profit, respectively, we follow Matsumura [13] and assume that the public firm maximizes a weighted average of social welfare and its own profit. Therefore, we define the objective function of the public firm, denoted by $V(\theta)$, as follows.

$$\begin{aligned} V(\theta) &\equiv (1-\theta)W(q_0,q_1,N) + \theta\pi_0(q_0,q_1,N) \\ &= (1-\theta)\left[a(q_0+q_1) - \frac{(q_0+q_1)^2}{2N} - \frac{1}{2}q_1^2 + \frac{t}{2}N^2 - N\bar{U}\right] \\ &+ \ \theta\left(a - \frac{q_0+q_1}{N}\right)q_0 - \frac{\alpha}{2}q_0^2 \end{aligned} \tag{18}$$

Let $V(\theta)$ represent the objective function of the public firm. Taking θ and q_1 as given and differentiating (18) with respect to q_0, the first-order condition for the maximization of the public firm is given by

$$\frac{\partial V}{\partial q_0} = a - \frac{q_0+q_1}{N} - \theta\frac{q_0}{N} - \alpha q_0 = 0. \tag{19}$$

Differentiating (16) with respect to q_1, the first-order condition for the private firm is given by

$$\frac{\partial \pi_1}{\partial q_1} = a - \frac{q_0+2q_1}{N} - q_1 = 0 \tag{20}$$

Solving (19) and (20) simultaneously, the equilibrium outputs of the public firm and the private firm are derived as follows.

$$q_0^* = \frac{aN(N+1)}{\alpha N^2 + (2\alpha+1+\theta)N + (1+2\theta)} \tag{21}$$

and

$$q_1^* = \frac{aN(\alpha N + \theta)}{\alpha N^2 + (2\alpha + 1 + \theta)N + (1 + 2\theta)} \tag{22}$$

From (21) and (22), total output of manufactured goods denoted by Q^* is given by

$$Q^* = \frac{aN\left[(\alpha + 1)N + (1 + \theta)\right]}{\alpha N^2 + (2\alpha + 1 + \theta)N + (1 + 2\theta)}. \tag{23}$$

Moreover, substituting (23) into (13), we derive the equilibrium price of manufactured goods as follows.

$$p^* = \frac{a[\alpha N^2 + (\alpha + \theta)N + \theta]}{\alpha N^2 + (2\alpha + 1 + \theta)N + (1 + 2\theta)}. \tag{24}$$

2.4 Existence and uniqueness of urban equilibrium

While the number of households in the city, N^*, was taken as given in the previous subsection, it is endogenously determined by (12), which characterizes the equilibrium condition of the urban structure. Specifically, the urban equilibrium is characterized by the condition that the indirect utility of households in the city equals the exogenously given utility level $\bar{U}$. Therefore, we examine the existence and uniqueness of the urban equilibrium. First, we show that the equilibrium number of households in the city is uniquely determined by combining the equilibrium conditions of the urban structure and the manufactured goods market. Substituting (24) into (12), the condition that jointly characterizes the equilibrium urban structure and the equilibrium in the manufactured goods market is given by

$$N^* = \frac{1}{t}\left[a^2\left(\frac{(\alpha + 1)N^* + (1 + \theta)}{\alpha(N^*)^2 + (2\alpha + 1 + \theta)N^* + (1 + 2\theta)}\right)^2 + 2\left(1 - \bar{U}\right)\right] \tag{25}$$

where $f(N)$ and $g(N)$ are defined as follows.

$$\begin{aligned} f(N) &\equiv (\alpha + 1)N + (1 + \theta) \\ g(N) &\equiv \alpha N^2 + (2\alpha + 1 + \theta)N + (1 + 2\theta) \end{aligned}$$

Moreover, we define the function $F(N)$ as

$$F(N) \equiv tN - \left[a^2\left(\frac{f(N)}{g(N)}\right)^2 + 2(1 - \bar{U})\right]. \tag{26}$$

Differentiating (26) with respect to N, we obtain

$$\frac{dF(N)}{dN} = t - 2a^2 \left(\frac{f(N)}{g(N)}\right) \frac{d}{dN} \left(\frac{f(N)}{g(N)}\right). \tag{27}$$

Next, taking account of $\theta \in [0,1]$ and $\alpha > 1$, we define $h(N)$ as $\frac{f(N)}{g(N)}$ and investigate the sign of $\frac{dh(N)}{dN}$. Differentiating $h(N)$ with respect to N, $\frac{dh(N)}{dN}$ is given by

$$\frac{dh(N)}{dN} = -\frac{\alpha(\alpha+1)N^2 + 2\alpha(\theta+1)N + \left(\alpha+\theta^2\right)}{[g(N)]^2} < 0 \tag{28}$$

When we assume $\alpha > 1$, $N \geq 0$, and $\theta \in [0,1]$, all terms on the right-hand side are negative. Consequently, the second term in (27) is negative, implying that the sign of (27) is positive and that $F(N)$ is a monotonically increasing function of N. Next, we examine the sign of $F(0)$ by substituting $N = 0$ into $F(N)$:

$$F(0) = -a^2 \left(\frac{1+\theta}{1+2\theta}\right)^2 - 2\left(1 - \bar{U}\right) < 0. \tag{29}$$

Since $F(N)$ is continuous and strictly increasing in $N \in [0, \infty)$, and $F(0) < 0$, there exists a unique $N > 0$ such that $F(N) = 0$ because of the Intermediate Value Theorem. Since the city boundary is given by $r_f = N$, it follows that the equilibrium city size r_f^* is uniquely determined. Therefore, we obtain the following proposition.

Proposition 1. *Under $\alpha > 1$ and $t > 0$, the equilibrium city population N^* and the equilibrium city boundary r_f^* are uniquely determined.*

3 Comparative statics

In the preceding sections, we have characterized household behavior, urban structure, firm behavior, and the existence and uniqueness of the urban equilibrium. In this section, we conduct comparative statics to examine how the degree of privatization of the public firm affects the equilibrium price of manufactured goods, urban structure, and social welfare.

3.1 Privatization of public firm and the equilibrium Price

Taking into account that both firms determine their outputs given the number of households in the city, we can examine the impact of the degree of privatization of the public firm, denoted by θ, on the equilibrium price of manufactured goods in a partial equilibrium framework where the number of households in the city is held fixed. By differentiating (24) with respect to θ, the impact of θ on the equilibrium price p^* is obtained as follows.

$$\frac{dp^*}{d\theta} = \frac{a(N+1)^2}{\Big[\alpha N^2 + (2\alpha + 1 + \theta)N + (1 + 2\theta)\Big]^2} > 0 \tag{30}$$

From (30), we see that an increase in the degree of privatization of the public firm raises the equilibrium price of manufactured goods. Therefore, we obtain the following lemma.

Lemma 1. *Under $a > 0$, $\alpha > 1$, $N \geq 0$, and $\theta \in [0, 1]$, a higher degree of privatization of the public firm increases the equilibrium price of manufactured goods p^*.*

This result shows that an increase in the degree of privatization of the public firm reduces output. Because the reduction in the public firm's output is larger than the increase in the private firm's output due to strategic substitutability, the total quantity of manufactured goods supplied to the market decreases. Consequently, an increase in degree of privatization (θ) raises the equilibrium price of manufactured goods. This result is consistent with the results obtained by Matsumura[13].

3.2 Privatization of public firm and the equilibrium urban structure

The number of households, representing the city size, is determined by the endogenous variables p and w and the parameters a, t, and $\bar{U}$. Recall that the endogenous variables p and w are determined by (24) and the normalization $w = 1$, respectively. Substituting (24) and $w = 1$ into (12), the equilibrium number of households, denoted by N^*, is determined as the value of N that satisfies (25). Applying the Implicit Function Theorem to (25), we obtain the following.

$$\frac{dN^*}{d\theta} = -\frac{2a^2\, f(N^*, \theta)\, (N^* + 1)^2}{t\, g(N^*, \theta)^3 + 2a^2\, f(N^*, \theta)\, K(N^*, \theta)} < 0, \tag{31}$$

where $f(N^*, \theta)$, $g(N^*, \theta)$, and $K(N^*, \theta)$ are defined as follows: $f(N^*, \theta) \equiv (\alpha + 1)N^* + (1 + \theta)$, $g(N^*, \theta) \equiv \alpha(N^*)^2 + (2\alpha + 1 + \theta)N^* + (1 + 2\theta)$, $K(N^*, \theta) \equiv \alpha^2(N^*)^2 + \alpha(N^*)^2 + 2\alpha\theta N^* + 2\alpha N^* + \alpha + \theta^2$. We see that the equilibrium number of households (i.e., the equilibrium city size) decreases as the degree of privatization of the public firm (θ) increases. Consequently, we obtain the following proposition.

Proposition 2. *Under $\alpha > 1$ and $t > 0$, an increase in the degree of privatization of the public firm reduces the equilibrium city population N^* and the equilibrium city boundary r_f^*.*

3.3 Privatization of public firm and the social welfare

Finally, we examine how the degree of privatization of the public firm affects equilibrium social welfare. Since p^* and N^* are uniquely determined, we consider how a change in θ, the degree of privatization of the public firm, affects equilibrium social welfare. Recall that the social welfare function is defined by (17). The effect of the degree of privatization of the public firm on social welfare can be divided into a direct effect and an indirect effect through changes in the city size. From Proposition 1, the equilibrium city size given by (31) is a decreasing function of θ. Therefore, equilibrium social welfare can be expressed as $W^*(\theta, N^*(\theta))$. Differentiating $W^*(\theta, N^*(\theta))$ with respect to θ, the impact of θ on W^* is given by

$$\frac{dW^*(\theta, N^*(\theta))}{d\theta} = \left.\frac{\partial W^*}{\partial \theta}\right|_{N=N^*(\theta)} + \frac{\partial W^*}{\partial N} \cdot \frac{dN^*}{d\theta} \tag{32}$$

The first and second terms on the right-hand side of (32) represent the direct and indirect effects of θ, respectively. First, we consider the direct impact of θ on $W^*(\theta, N^*(\theta))$. From (32), the sign of the direct effect cannot be determined unambiguously. The sign of the direct effect reflects a trade-off between the distortion-correcting effect due to reduced output of the inefficient public firm and the cost-increasing effect associated with expanded output by the private firm as a strategic substitute. [1]

Next, we examine the indirect effect of θ on W^*. Differentiating $W^*(\theta, N^*(\theta))$ with respect to N^*, the indirect effect corresponds to the second term on the right-hand side of (32). Since $\frac{dN^*}{d\theta}$ is negative according to (31), the sign of the second term depends on the sign of $\frac{\partial W^*}{\partial N^*}$. The sign of $\frac{\partial W^*}{\partial N^*}$ is governed by the interaction among the market-size effect for manufactured goods induced by a larger city size N^*, the increase in aggregate land rent revenue associated with urban expansion, and the marginal reservation utility effect.

[1] See Appendix A for the detailed derivations and decomposition of the direct and indirect effects.

Hence, the sign is in general ambiguous. Therefore, we derive the following proposition:

Proposition 3. *When equilibrium city size denoted by $N^*(\theta)$ and equilibrium social welfare denoted by $W^*(\theta)$ under $a > 0$, $t > 0$, $\alpha > 1$, $\theta \in [0,1]$, $\frac{dW^*(\theta)}{d\theta}$ is given by*

$$\frac{dW^*(\theta)}{d\theta} = \left.\frac{\partial W}{\partial \theta}\right|_{N=N^*(\theta)} + \left.\frac{\partial W}{\partial N}\right|_{q_0=q_0^*,\, q_1=q_1^*} \frac{dN^*(\theta)}{d\theta} \gtreqless 0$$

. Therefore, This result means that privatization of the public firm does not necessarily improve social welfare.

Since Matsumura [13] does not incorporate an urban structure, he shows that welfare under partial privatization of a public firm is higher than under full nationalization. In contrast, the model developed in this paper, which embeds a monocentric city structure into an open-city framework, implies that partial privatization does not necessarily maximize social welfare. Therefore, the optimality of partial privatization is conditional in our setting.

3.4 Numerical example

Lemmas 1 and Propositions 1 and 3 imply that an increase in θ, representing the degree of privatization of the public firm, raises the equilibrium price of manufactured goods p^*, reduces the equilibrium city size $N^*(= r_f^*)$, and does not necessarily reduce equilibrium social welfare W^*. In particular, the impact of θ on equilibrium social welfare is in general ambiguous, as it depends on the underlying parameter values. If the social welfare function is monotonic in θ over the interval $\theta \in [0,1]$, the welfare-maximizing degree of privatization of the public firm is a corner solution. When $\theta = 0\,(1)$, the welfare-maximizing outcome is full nationalization (full privatization). When the equilibrium social welfare function has an interior maximum at some $\theta \in (0,1)$, partial privatization is optimal.

Figure 1 illustrates that the equilibrium social welfare function is decreasing in θ under the parameter values $a = 2$, $\alpha = 2$, $t = 0.5$, $\bar{U} = 0.7$, and $w = 1$. In this case, the optimal degree of privatization of the public firm is $\theta = 0$, that is, full nationalization.

Next, Figure 2 illustrates that the equilibrium social welfare function is monotonically increasing in θ under the parameter values $a = 0.403166$, $\alpha = 3.521907$, $t = 0.646375$, and $\bar{U} = 0.7$. In this case, the welfare-maximizing

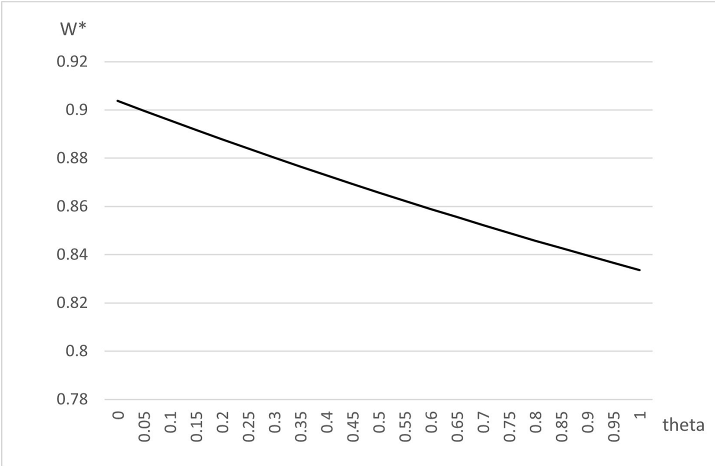

Figure 1: The relationship θ and $W^*(\theta = 0$ is optimal)

degree of privatization of the public firm is $\theta = 1$, corresponding to full privatization.

Finally, Figure 3 illustrates that the equilibrium social welfare function has has an interior maximum at some $\theta \in (0, 1)$ under the parameter values $a = 0.638487$, $\alpha = 2.757913$, $t = 0.518367$, $\bar{U} = 0.7$, $w = 1$. Since the welfare-maximizing degree of privatization of the public firm lies in the interior of the interval $(0, 1)$, the optimal policy is partial privatization, consistent with Matsumura [13]. From Figure1-3, we see that the optimal degree privatization of the public firm depends on the underlying parameter values. Therefore, in our model incorporating urban structure, the optimality of partial privatization established in Matsumura [13] is no longer general.

4 Conclusion

This paper extends the model of Naito[18] , which analyzes a mixed oligopoly market in which a public firm and a private firm compete within the framework of a monocentric open-city model originally developed by Alonso[3] and subsequent studies, by introducing the possibility of partial privatization of the public firm as considered in Matsumura[13]. We then examine how the degree of privatization of the public firm affects not only the price of the good but also the equilibrium city size and social welfare.

Matsumura [13] treats the market size as an exogenous variable and shows that partial privatization of the public firm is optimal for maximiz-

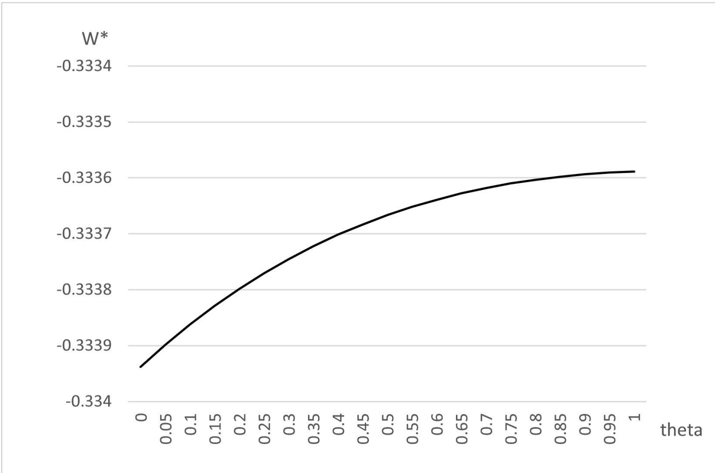

Figure 2: The relationship θ and $W^*(\theta = 1$ is optimal)

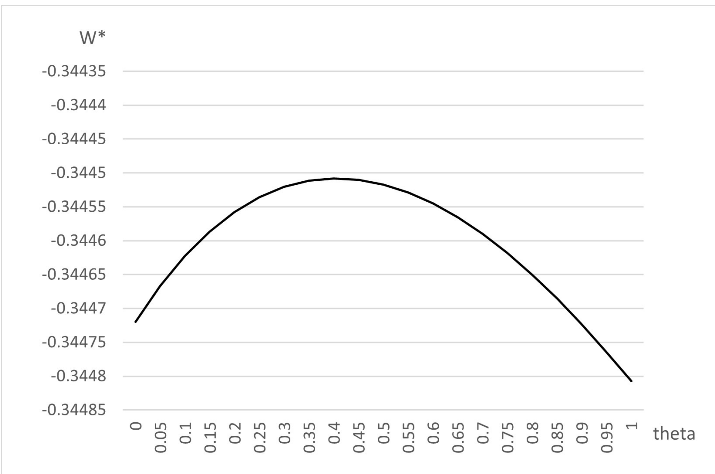

Figure 3: The relationship θ and $W^*(\theta \in (0,1)0$ is optimal)

ing equilibrium welfare. We incorporate an open monocentric city structure into the framework of Matsumura [13] and investigate whether the conclusion derived there continues to hold in our model. Our analysis shows that the optimality of partial privatization established in Matsumura [13] does not always hold, because a change in the degree of privatization of the public firm has not only a direct effect on the production sector of manufactured goods but also an indirect effect through the equilibrium city size, which is determined endogenously. Regarding to the optimal degree of privation, it depends on parameters values such as a, t, $\bar{U}$. Therefore, our analysis demonstrates that, depending on the parameter configuration, full nationalization or full privatization may be socially optimal, and the optimality of partial privatization established in Matsumura [13] does not hold in general.

For analytical simplicity, we impose the following assumptions. First, we assume that all households in a city consume one unit of land inelastically. Most monocentric city models incorporate lot size consumption into the utility function, and it is possible to allow lot size to vary with distance from the CBD. In this paper, we deliberately omit this feature so as to explicitly identify the effect of the degree of privatization of the public firm on the equilibrium city size. Second, since we use a small open city model in our paper, we consider the utility of households as reservation utility, which is given exogenously. Therefore, this utility is not determined endogenously. In this regard, it is possible to incorporate a dual-economy structure, as in Lewis [12], Harris and Todaro [7], and Naito [17], or a new economic geography framework, as in Abdel-Rahman [1] and Krugman [10]. Although we focuses on a duopoly in which a public firm and a private firm compete, the framework can be extended to an oligopolistic market structure, as considered in De Fraja and Delbono [5] and Matsumura [13], as well as to models with product differentiation. Introducing these elements and reconsidering the effects of public firm privatization would be an interesting avenue for future research. We leave these issues for future work.

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A The effects of privatization of public firm on social welfare

A.1 Direct effect

Here, we examine the effect of a change in θ on equilibrium welfare, holding N^* fixed. From (21), (22), (23), (17) is given by

$$W^* = aQ^* - \frac{(Q^*)^2}{2N} - \frac{\alpha}{2}(q_0^*)^2 - \frac{1}{2}(q_1^*)^2 + \frac{t}{2}N^2 - N\bar{U}. \tag{33}$$

Differentiating (33) with respect to θ, we obtain the following:

$$\left.\frac{\partial W^*}{\partial \theta}\right|_{N=N^*} = a\frac{\partial Q^*}{\partial \theta} - \frac{Q^*}{N}\frac{\partial Q^*}{\partial \theta} - \alpha q_0^* \cdot \frac{\partial q_0^*}{\partial \theta} - q_1^*\frac{\partial q_1^*}{\partial \theta} \tag{34}$$

Differentiating (21) and (22) with respect to θ, respectively, we obtain as follows.

$$\frac{\partial q_0^*}{\partial \theta} = -\frac{aN(N+1)(N+2)}{[\alpha N^2 + (2\alpha + 1 + \theta)N + (1 + 2\theta)]^2} < 0$$

and

$$\frac{\partial q_1^*}{\partial \theta} = \frac{aN\big(\alpha N^2 + \alpha N + N + 1\big)}{\Big[\alpha N^2 + (2\alpha + 1 + \theta)N + (1 + 2\theta)\Big]^2} > 0.$$

Since the sum of the first and second terms on the right-hand side of (34) the impact of $\theta \int_0^{Q^*} p(q)\, dq$, which is the total benefit defined by the area under the inverse demand curve, we know that an increase in θ decreases the total benefit. Next, taking into account that $\frac{dN^*}{d\theta} < 0$, we see that an increase in θ reduces total benefit. Since $\frac{\partial q_0^*}{\partial \theta} < 0$, the first term on the right-hand side of (34) is positive, reflecting the cost-reduction effect associated with the decline in output of the inefficient public firm. The fourth term reflects the increase in costs associated with the rise in the private firm's output induced by the reduction in production of the public firm under strategic substitutability. Therefore, the direct effect is determined by the trade-off among these three effects, and thus the sign of (34) is generally not determined uniquely.

A.2 Indirect effect

Next, we consider the indirect effect. Differentiating (33) with respect to N^*, $\frac{\partial W^*}{\partial N^*}$ is given by

$$\frac{\partial W^*}{\partial N^*} \cdot \frac{\partial N^*}{\partial \theta} = \left[\frac{(q_0 + q_1)^2}{2(N^*)^2} + tN^* - \bar{U}\right] \cdot \frac{\partial N^*}{\partial \theta}. \tag{35}$$

In (35), the first term inside the square brackets on the right-hand side represents the market-size effect for manufactured goods resulting from an increase in the city population N^*. The second term captures the increase in total land rent due to the expansion of the city size, and the third term represents the marginal reservation utility. While the first and second terms inside the square brackets are negative, the third term is positive. Therefore, the sign of (35) cannot be determined unambiguously.